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Sphere Packings and Hyperbolic Reflection Groups

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0. INTRODUCTION

The well-known "Apollonian" packing of circles is obtained by drawing within a given circle ω_1 three further circles ω_2, ω_3, ω_4 which touch each other, as well as ω_1, and proceeding in this manner to fill up the interstices (Fig. 1). The resultant packing is clearly maximal, but in fact enjoys the stronger property of being complete, in the sense that the residual set of points in ω_1 which are outside all of the constructed circles has measure zero.

An algebraic method for generating the same circles is to first construct circles $e_1,..., e_4$ orthogonal to $\omega_1,..., \omega_4$ and then successively apply inversions $s_1,..., s_4$ in $e_1,..., e_4$ to the original circles $\omega_1,..., \omega_4$. If W is the group generated by $s_1,..., s_4$, the packing is thus the set

$$\Omega = \bigcup_{i=1}^{4} W\omega_i.$$

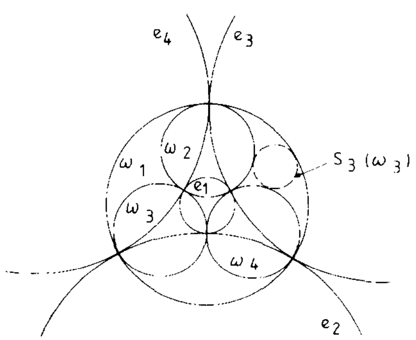

FIGURE 1

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Boyd [2, 3] has studied several analogous examples of such packings in up to nine dimensions.

Using the classical isomorphism between the conformal group in $(N-2)$-dimensional inversive space and the "isochronous" Lorentz group of an N-dimensional Minkowski space, and the resultant correspondence between spheres and vectors, one can regard Ω as the set of vectors obtained by letting a reflection group W act on its fundamental weights ω_i. In general, only some of the ω_i will correspond to "real" spheres, namely, those for which $(\omega_i, \omega_i) > 0$, and one must restrict oneself to the "real subset Ω_r of Ω.

Starting with a Coxeter group W which is naturally realisable in a Minkowski space, one can ask under what circumstances Ω_r is a packing. It turns out that this happens precisely when the Coxeter graph Γ of W is of "level $\leqslant 2$," i.e., if the deletion of any two vertices from Γ leaves a positive or a euclidean graph. Such graphs are an extension of the well-known "hyperbolic" graphs, which would be called of "level 1" in our terminology. Since Ω_r is empty for level 1 graphs, only level 2 graphs are of interest in this context. They have at most 11 vertices and thus produce examples of sphere packings in euclidean spaces of dimensions only up to 9. A complete classification can therefore be given, especially as distinct graphs often lead to the same packing.

Under a certain plausible hypothesis, Ω_r can be proved to be a maximal packing. It is probably then in fact complete, but Boyd's proof of this for his examples in [3] appears to be inadequate. The examples themselves can all be obtained in the manner indicated in this paper.

1. Coxeter Groups

Several results proved in this section occur as exercises in [1] and elsewhere; we include them for the sake of completeness.

Let $W = \langle s_1, ..., s_N \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ be an abstract Coxeter group and Γ its Coxeter graph [1]. If V is a real vector space with basis $\{e_1, ..., e_N\}$, define an inner product in V by

$$\begin{aligned} (e_i, e_j) &= -\cos(\pi/m_{ij}) \qquad \text{if} \quad m_{ij} < \infty \\ &= -c_{ij} \qquad\qquad\quad\;\; \text{if} \quad m_{ij} = \infty, \end{aligned} \tag{1.1}$$

where the c_{ij} are arbitrary real numbers $\geqslant 1$ satisfying $c_{ij} = c_{ji}$. When $c_{ij} > 1$, we adopt Vinberg's convention [9] of replacing the solid edge between vertices i and j in Γ by a dotted edge marked with c_{ij}. To realise W as a geometric group generated by reflections, let s_i act on V by $s_i \cdot v = v - 2(v, e_i)\, e_i$. The cases when $(v, v) \geqslant 0$ for all $v \in V$ are of course well known; the corresponding graphs Γ are called *positive* or *euclidean* according to whether or not $(v, v) > 0$ for all $v \neq 0$. If the inner product (1.1) is of signature $(N-1, 1)$, and thus nonsingular, Γ is said to be *hyperbolic*. The structure of hyperbolic graphs is not known, except in special cases [6].

We say that Γ is of *level* $\leqslant l$ if the deletion of any l vertices from Γ leaves a positive or euclidean graph. If Γ is not also of level $\leqslant l-1$, then l is called the level of Γ. A graph of level 1 is clearly connected; such graphs are known to exist only for $2 \leqslant N \leqslant 10$. A complete list can be found for example in [5], to which should be added all graphs of the form $\circ\overset{c}{-}\circ$. Consequently, graphs of level l can exist only for $l+1 \leqslant N \leqslant l+9$. A graph of level 2 is either connected, or else obtained by adding an isolated vertex to a graph of level 1.

PROPOSITION 1.1. *If Γ is a connected graph of level l, the deletion of any $l+1$ vertices from Γ always leaves a positive graph.*

Proof. Since the deletion of any vertex from a euclidean graph leaves a positive graph, the statement is true for $l=0$. In general, suppose that the deletion of l vertices $i_1, \ldots, i_l$ from Γ leaves a euclidean graph Γ'. If Γ' is not connected, let Γ'' be a connected euclidean component of Γ' and j a vertex in $\Gamma' - \Gamma''$. Since Γ is connected, at least one of the vertices $i_1, \ldots, i_l$, say i_1, is joined to an element of Γ''. Then $\Gamma - \{j, i_2, \ldots, i_l\}$ contains a euclidean component which includes Γ'' and i_1, contradicting our initial observation. Thus Γ' is connected and therefore the deletion of a further vertex from Γ leaves a positive graph. ∎

We say that a graph Γ of level l is *strict* if the deletion of any l vertices from Γ leaves a positive graph. Strict graphs of level 1 are usually called "compact" and exist only for $2 \leqslant N \leqslant 5$. Consequently, strict graphs of level l can exist only for $l+1 \leqslant N \leqslant l+4$.

Let V^* be the dual space of V, $\{\omega_1, \ldots, \omega_N\}$ the dual basis of $\{e_1, \ldots, e_N\}$ and let W act on V^* by the contragredient rule $\langle v, w \cdot v^* \rangle = \langle w^{-1} \cdot v, v^* \rangle$, where $v \in V$, $v^* \in V^*$ and $w \in W$. The ω_i are called *fundamental weights*, whereas elements of the set

$$\Omega = \bigcup_{i=1}^{N} W\omega_i$$

are simply called *weights*. The convex closure of Ω in V^* is the *Tits cone* $\mathcal{T}$, which can also be described as follows. Let $\mathcal{C}$ be the cone of all $v^* \in V^*$ which satisfy $\langle e_i, v^* \rangle \geqslant 0$ for $i = 1, \ldots, N$, and $\mathcal{C}^0$ the interior of $\mathcal{C}$. Then $\bigcup_{w \in W} w(\mathcal{C})$ is convex [1] and therefore equal to $\mathcal{T}$. The *polar* cone of $\mathcal{T}$ is defined as

$$\mathcal{T}^p = \{v \in V \mid \langle v, t^* \rangle \leqslant 0 \text{ for all } t^* \in \mathcal{T}\}.$$

Both $\mathscr{C}$ and $\mathscr{C}^p$ are invariant under W. It is known in general that the double polar $\mathscr{C}^{pp}$ of any convex cone $\mathscr{C}$ is equal to the topological closure $\bar{\mathscr{C}}$ of $\mathscr{C}$ [7].

PROPOSITION 1.2. *Every element $v \in \mathscr{C}^p$ satisfies $(v, v) \leqslant 0$.*

Proof. Suppose $v \in \mathscr{C}^p$ and $(v, v) > 0$. Define $\phi(v) = \sum_i \langle v, \omega_i \rangle$; clearly $\phi(v) < 0$. The inequality

$$(v, e_i) \leqslant (v, v)/\phi(v) \qquad (*)$$

must hold for at least one index i; otherwise, writing $v = \sum_i \langle v, \omega_i \rangle e_i$, we conclude that

$$(v, v) = \sum_i \langle v, \omega_i \rangle (v, e_i) < (v, v)$$

since all $\langle v, \omega_i \rangle \leqslant 0$ and at least one is $\neq 0$.

Choosing an index i for which $(*)$ is true, we observe that

$$0 > \phi(s_i(v)) = \phi(v) - 2(v, e_i) \geqslant \phi(v) - 2(v, v)/\phi(v).$$

Since $s_i(v) \in \mathscr{C}^p$, the argument may be repeated to find an index j such that

$$0 > \phi(s_j s_i(v)) \geqslant \phi(s_i(v)) - 2(v, v)/\phi(s_i(v))$$
$$\geqslant \phi(v) - 4(v, v)/\phi(v).$$

At the nth stage we have

$$0 > \phi(v) - 2n(v, v)/\phi(v),$$

which is a contradiction once $n > \phi(v)^2/2(v, v)$. ∎

When the inner product (1.1) is nonsingular, we shall identify V^* with V by its aid and thus regard $\mathscr{C}$ as a subset of V.

Observe that $\mathscr{C}^p$ is always a *pointed* cone, i.e., $\mathscr{C}^p \cap (-\mathscr{C}^p) = 0$ since an element v in the intersection must satisfy $\langle v, \omega_i \rangle = 0$ for all i. It follows from Proposition 1.2 that when Γ is positive, $\mathscr{C}^p = \{0\}$, so that $\mathscr{C} = \bar{\mathscr{C}} = V$, whereas if Γ is euclidean and connected, $\mathscr{C}^p$ is a half-line, so that $\bar{\mathscr{C}}$ is a half-space. When Γ is hyperbolic, the cone

$$\{v \in V \mid (v, v) \leqslant 0\} \qquad (1.2)$$

has two connected components (after deleting 0), which are also the equivalence classes for the relation

$$u \sim v \Leftrightarrow (u, v) \leqslant 0.$$

It follows that each of these components (with 0 added) is self-polar. The cone $\mathscr{C}^p$, being pointed, must be contained in one of the components. Taking polars, this implies

COROLLARY 1.3. *If Γ is hyperbolic, $\mathscr{C}$ contains a component of the cone* (1.2).

An element $u \in V$ is *real* if $(u, u) > 0$. Distinct elements u, $v \in V$ are *disjoint* if $(u, v) \leqslant 0$ and the restriction of $(\cdot, \cdot)$ to the subspace spanned by u and v is *not* positive definite, i.e., if $(u, v)^2 \geqslant (u, u)(v, v)$.

If $v = \sum v_i e_i$, let $v_+ = \sum_{v_i > 0} v_i e_i$ and $v_- = \sum_{v_i < 0} v_i e_i$, then $(v_+, v_-) \geqslant 0$ and, if Γ is connected, equality is possible only if $v = v_+$ or $v = v_-$.

PROPOSITION 1.4. *A graph of level* 1 *is hyperbolic. All fundamental weights are pairwise disjoint and none are real.*

Proof. Suppose $v \in V$ is such that $(v, v) \leqslant 0$. If $v \neq v_+$ or v_-, we have $(v_+, v_+) < 0$ or $(v_-, v_-) < 0$, both of which contradict the fact that Γ is of level 1. Furthermore, if $(v, v) < 0$, all v_i are $\neq 0$, whereas if $(v, v) = 0$, at most one $v_i = 0$, in view of Proposition 1.1.

If Γ is not hyperbolic, V contains nonzero orthogonal vectors u and v such that $(u, u) < 0$ and $(v, v) = 0$. If $a = v_i/u_i$ for some $v_i \neq 0$, the vector $u' = au - v$ satisfies $(u', u') < 0$ and $u'_i = 0$, a contradiction.

Since the subspace of V orthogonal to ω_i is spanned by $\{e_j \mid j \neq i\}$, we must have $(\omega_i, \omega_i) \leqslant 0$ for all i, since otherwise $\Gamma - i$ would be hyperbolic. From

$$\omega_i = (\omega_i, \omega_i)\, e_i + \sum_{j \neq i} (\omega_i, \omega_j)\, e_j \tag{1.3}$$

follows the equation

$$1 = (e_i, \omega_i) = (\omega_i, \omega_i) + \sum_{j \neq i} (\omega_i, \omega_j)(e_i, e_j). \tag{1.4}$$

If $(\omega_i, \omega_i) < 0$, one sees from (1.3) that all (ω_i, ω_j) are <0; when $(\omega_i, \omega_i) = 0$, the numbers (ω_i, ω_j) must all be of the same sign, namely, negative to satisfy (1.4). By Proposition 1.1, any two fundamental weights must span a hyperbolic plane, and are thus disjoint. ∎

Combining this Proposition with Corollary 1.3, we deduce

COROLLARY 1.5. *If Γ is hyperbolic of level* 1, *$\mathscr{C}$ is equal to a component of the cone* (1.2).

PROPOSITION 1.6. *A graph of level* 2 *is hyperbolic. All fundamental weights are pairwise disjoint; ω_i is real whenever $\Gamma - i$ is of level* 1, *in which case we have $(\omega_i, \omega_i) \leqslant 1$.*

Proof. We may assume that Γ is connected. Suppose $v \in V$ is such that $(v, v) \leqslant 0$. If $v \neq v_+$ or v_-, either (v_+, v_+) or (v_-, v_-) is <0, which can happen only if all v_i are $\neq 0$ and precisely one v_j is of a different sign from the rest. Consequently, if $(v, v) < 0$, then apart from at most one index j, the v_i are of the same sign and $\neq 0$. When $(v, v) = 0$, the same conclusion holds, except that it may also happen that two of the v_i are zero, while the rest are of the same sign and $\neq 0$.

If Γ is not hyperbolic, V contains nonzero orthogonal vectors u and v such that $(u, u) < 0$ and $(v, v) = 0$. Suppose that Γ contains at least four vertices. Let $a = v_i/u_i$ for some i such that u_i and v_i are $\neq 0$; then $u' = au - v$ can replace u, but now $u'_i = 0$. We may thus assume initially that $u_i = 0$ and $u_j > 0$ for $j \neq i$. If at least two of the numbers v_j/u_j $(j \neq i)$ are nonzero, choose $a = v_k/u_k \neq 0$ such that $a \leqslant v_m/u_m$ for some $m \neq k$, but $a \geqslant v_j/u_j$ for all $j \neq i, k, m$, a contradiction since $(u', u') < 0$. If only one v_j/u_j is nonzero for $j \neq i$, then $m_{ij} = \infty$, $(e_i, e_j) = -1$ and v can be taken as $e_i + e_j$. Since e_j is then orthogonal to v, $(v, u) = 0$ implies that $\{i, j\}$ is disconnected from the rest of Γ, again a contradiction. The case when Γ has three vertices can be settled by a simple direct argument, which we omit.

If $(\omega_i, \omega_i) \leqslant 0$, we see from (1.3) and (1.4) that $(\omega_i, \omega_j) \leqslant 0$, with the possible exception of one index $j \neq i$. However, taking the inner product of both sides of (1.3) with e_j shows this to be impossible. Clearly $(\omega_i, \omega_i) > 0$ if and only if $\Gamma - i$ is hyperbolic and hence of level 1. The orthogonal projection $e'_i = e_i - (\omega_i, \omega_i)^{-1} \omega_i$ of e_i on $(\mathbb{R}\omega_i)^\perp$ satisfies $(e'_i, e_j) \leqslant 0$ for all $j \neq i$ and consequently belongs to the cone $-\mathscr{C}$ for $\Gamma - i$. Therefore $(e'_i, e'_i) = 1 - (\omega_i, \omega_i)^{-1} \leqslant 0$, i.e., $(\omega_i, \omega_i) \leqslant 1$, and $(e'_i, \omega'_j) = (e'_i, \omega_j) = -(\omega_i, \omega_i)^{-1}(\omega_i, \omega_j) \geqslant 0$ for $j \neq i$ by Proposition 1.4, so that $(\omega_i, \omega_j) \leqslant 0$. Since $\Gamma - i - j$ is positive or euclidean for all distinct i, j, any two fundamental weights ω_i, ω_j do not span a positive definite subspace and are therefore disjoint. ∎

Recall from [1] the following basic:

PROPOSITION 1.7. *If $v \in \mathscr{C}^0$ and $w \in W$, then for all i, $l(s_i w) > l(w)$ if and only if $(w(v), e_i) > 0$ (where $l(w)$ denotes the length of w).*

COROLLARY 1.8. *If $v \in \mathbb{C}$, then $(w(v), e_i) \geqslant 0$ if $l(s_i w) > l(w)$ and $(w(v), e_i) \leqslant 0$ if $l(s_i w) < l(w)$.*

Using this, we can establish

THEOREM 1.9. *The following are equivalent:*

(a) *Γ is of level 1 or 2;*

(b) *Γ is hyperbolic and any two weights are disjoint.*

Proof. (a) ⇒ (b): it suffices to prove that if $\omega_i \neq w(\omega_j)$, then

$$(\omega_i, w(\omega_j)) \leqslant 0 \tag{1.5}$$

and

$$(\omega_i, w(\omega_j))^2 \geqslant (\omega_i, \omega_i)(\omega_j, \omega_j). \tag{1.6}$$

We proceed by induction on $l(w)$, the case $w = 1$ being already known. One can assume that $l(s_k w) > l(w)$ for all $k \neq i$, since otherwise w can be replaced by $s_k w$ in (1.5) and (1.6), as $s_k(\omega_i) = \omega_i$. Therefore $w = s_i w'$ with $l(w) > l(w')$ and

$$\begin{aligned}(\omega_i, w(\omega_j)) &= (s_i(\omega_i), w'(\omega_j)) \\ &= (\omega_i, w'(\omega_j)) - 2(e_i, w'(\omega_j)).\end{aligned}$$

If $\omega_i \neq w'(\omega_j)$, we have $(\omega_i, w'(\omega_j)) \leqslant 0$ by the inductive hypothesis and $(e_i, w'(\omega_j)) \geqslant 0$ by Corollary 1.8, proving (1.5). If $\omega_i = w'(\omega_j)$, then $s_i(\omega_i) = w(\omega_j)$ and

$$\begin{aligned}(\omega_i, s_i(\omega_i)) &= (\omega_i, \omega_i) - 2 \\ &\leqslant 0\end{aligned}$$

by Proposition 1.6. To establish (1.6), suppose to the contrary that ω_i and $w(\omega_j)$ span a positive definite subspace; then $\Gamma - i$ must be of level 1. If v denotes the projection of $w(\omega_j)$ on $(\mathbb{R}\omega_i)^\perp$, then the subspace orthogonal to v in $(\mathbb{R}\omega_i)^\perp$ must be hyperbolic, so that $(v, v) > 0$. However, for all $k \neq i$, we have $(v, e_k) = (w(\omega_j), e_k) \geqslant 0$ since $l(s_k w) > l(w)$. Therefore v belongs to the cone $\mathscr{C}$ for $\Gamma - i$ and hence satisfies $(v, v) \leqslant 0$, a contradiction.

(b) ⇒ (a): since any two fundamental weights ω_i and ω_j do not span a positive definite subspace, the orthogonal complement of this subspace must be positive or euclidean, as Γ is hyperbolic. Therefore Γ is of level 1 or 2. ∎

Another consequence of Proposition 1.7 is

PROPOSITION 1.10. *If $u, v \in \mathscr{C}$ are such that $w(u) = v$ for some $w \in W$, then $u = v$ and w is a product of reflections s_i for which $(v, e_i) = 0$.*

Therefore "orbits" $W\omega_i$ and $W\omega_j$ are disjoint if $i \neq j$, and the elements of $W\omega_i$ correspond to left cosets of W^i in W, where W^i is the subgroup generated by all s_k with $k \neq i$.

The "exchange property" in W [1] implies that each left coset of W^i in W contains a *unique* element w of minimal length, characterised by the property that

$$l(ws_k) > l(w) \qquad \text{for all} \quad k \neq i. \tag{1.7}$$

An algorithm for constructing such elements w proceeds as follows. Suppose we know all minimal w of length n and that $w(\omega_i)$ has been expressed in terms of the basis $\omega_1,..., \omega_N$:

$$w(\omega_i) = \sum_k x_k \omega_k,$$

where $x_k = (w(\omega_i), e_k)$. If $x_j < 0$, then $l(s_j w) < l(w)$ by Corollary 1.8. If $x_j = 0$, then $s_j w(\omega_i) = w(\omega_i)$, so that $s_j w \in wW^i$. However, if $x_j > 0$, then $l(s_j w) > l(w)$ and, furthermore, we must have $l(s_j w s_k) > l(s_j w)$ for all $k \neq i$ since otherwise we would have either $s_j w s_k = w$ and hence $x_j = 0$, or $l(ws_k) < l(w)$, contradicting (1.7). It follows that whenever $x_j > 0$, $s_j w$ is a minimal element of length $n + 1$; *however*, it may happen that $s_j w = s_k w'$ for distinct minimal w, w' and distinct j, k.

As was pointed out by the referee, whenever Γ is connected, W acts faithfully on each coset space W/W^i and thus on each orbit $W\omega_i$. This follows for instance from Proposition 12.15 of [8]. In particular, we conclude that an orbit $W\omega_i$ can be finite only if W itself is finite.

2. The Fundamental Isomorphism

Let V be a real vector space of dimension $N \geqslant 3$, with a nonsingular symmetric bilinear form of signature $(N-1, 1)$. The orthogonal group $O(V)$ has a subgroup $O_i(V)$ of index 2 consisting of all "isochronous" elements, i.e., those which preserve the components of the cone (1.2). If $k \neq 0$ is a nonisotropic element of V, the reflection s_k belongs to $O_i(V)$ whenever $(k, k) > 0$; every element of $O_i(V)$ is the product of at most N such reflections.

Choose an element $p \in V$ for which $(p, p) = -1$. Then $(\mathbb{R}p)^{\perp}$ is positive definite and the hyperplane

$$(\mathbb{R}p)^{\perp} + p = \{v \in V \mid (v, p) = -1\} \tag{2.1}$$

can be regarded as a euclidean space of dimension $N-1$. Let $\mathscr{H}$ be the intersection of this hyperplane with the cone (1.2) and $\mathscr{H}^b$ its boundary. Every element of $\mathscr{H}$ can be written as $x + p$, with $(x, x) \leqslant 1$, so that $\mathscr{H}$ is the unit ball in (2.1) centered on p. The group $O_i(V)$ acts faithfully on the set of isotropic lines in V and consequently on $\mathscr{H}^b$. Now choose a point $q \in \mathscr{H}^b$, let E be the hyperplane in (2.1) orthogonal to $q - p$ and σ the steregraphic projection of $\mathscr{H}^b$ on $\tilde{E} = E \cup \{\infty\}$ with center q. Then E is a euclidean space of dimension $N-2$ and conjugation by σ induces a faithful action of $O_i(V)$ on $\tilde{E}$.

Consider the effect of a reflection s_k, with (k, k) normalised to be 1. Such a vector k can be written uniquely in one of the forms

$$k = \rho^{-1}(a + p + \tfrac{1}{2}((a, a) - 1 - \rho^2)q), \tag{2.2}$$

or

$$k = n + \alpha q, \tag{2.3}$$

where a and n are vectors in $E - p$ (and thus orthogonal to p and q), with $(n, n) = 1$, while $\rho \neq 0$ and α are real numbers. The two cases are distinguished by whether $(k, q) = -\rho^{-1} \neq 0$ or $(k, q) = 0$. A calculation shows that in case (2.2), an element $y + p \in E$, with $y \neq a$, is mapped to

$$a + p + \rho^2(y - a)/(y - a, y - a),$$

the inversion of $y + p$ in a sphere with center $a + p$ and radius $|\rho|$, while $a + p$ is interchanged with ∞. In case (2.3), $y + p$ is mapped to its reflection in the hyperplane of E with the equation

$$(n, v - p) = \alpha, \tag{2.4}$$

while ∞ is left fixed.

The conformal group of $\tilde{E}$ is, by definition, generated by inversions in all spheres and reflections in all hyperplanes of E. The action of $O_l(V)$ on $\tilde{E}$ therefore induces an isomorphism between $O_l(V)$ and this group.

A point $y + p \in E$ corresponds by σ to the point $2q_y/(1 + (y, y))$ in $\mathscr{H}^b$, where

$$q_y = y + p + \tfrac{1}{2}((y, y) - 1)q, \tag{2.5}$$

while ∞ corresponds to $q_\infty = q$. To each vector $k \in V$ satisfying $(k, k) = 1$, we associate the set

$$S_k = \{ y + p \in E \mid (k, q_y) \geqslant 0\} \tag{2.6}$$

in E, with ∞ added to S_k whenever $(k, q) \geqslant 0$. If k is of type (2.2),

$$(k, q_y) = (\rho^2 - (y - a, y - a))/2\rho,$$

so that S_k is the ball with center $a + p$ and radius $|\rho|$ if $(k, q) < 0$, or the complement of the interior of this ball if $(k, q) > 0$. In case (2.3), $(k, q_y) = (n, y) - \alpha$ and S_k is that half-space of $\tilde{E}$ determined by the hyperplane (2.4) to which n "points."

The number (k_1, k_2) is sometimes called the "separation" between S_{k_1} and S_{k_2}. If, for example, both k_1 and k_2 are of type (2.2),

$$(k_1, k_2) = (\rho_1^2 + \rho_2^2 - (a_1 - a_2, a_1 - a_2))/2\rho_1\rho_2.$$

Thus if $(k_1, q) < 0$ and $(k_2, q) < 0$, the inequality $(k_1, k_2) \leqslant -1$ is equivalent to saying that the balls S_{k_1} and S_{k_2} have at most one point in common (when $(k_1, k_2) = -1$).

3. Packings

In the context of Section 2, a nonempty subset $\mathscr{P}$ of V such that $(k, k) = 1$ for all $k \in \mathscr{P}$ and $(k, k') \leqslant -1$ for all $k \neq k'$ in $\mathscr{P}$ is called a *packing*. A set of real, pairwise disjoint elements of V can always be normalised to produce a packing. Packings of the form $\{k, -k\}$ are called *trivial*; any two elements k, k' of a nontrivial packing are independent, and $\alpha k + k'$ is isotropic for a unique $\alpha \geqslant 1$.

For each $k \in V$ such that $(k, k) = 1$, form the "spherical cap"

$$C_k = \{v \in \mathscr{H} \mid (v, k) \geqslant 0\}; \tag{3.1}$$

then $S_k = \sigma(C_k \cap \mathscr{H}^b)$ in the notation of Section 2.

Proposition 3.1. *If $\mathscr{P}$ is a nonempty subset of V, the following statements are equivalent*:

(a) *$\mathscr{P}$ is a nontrivial packing*;

(b) *for either $\mathscr{P}$ or $-\mathscr{P}$, any two caps C_k and $C_{k'}$ have at most one point in common* (*when* $(k, k') = -1$).

Proof. (a) $\Rightarrow$ (b): We first observe that if $k \neq k'$, $v \in C_k \cap C_{k'}$, and $v' \in C_{-k} \cap C_{-k'}$, then $(v, (k + k')) \geqslant 0$, $(v', k + k') \leqslant 0$ and $(k + k', k + k') \leqslant 0$. Since $v \sim v'$, this is possible only if $(k, k') = -1$ and either v or v' is a multiple of $k + k'$. Hence one of the sets $C_k \cap C_{k'}$, $C_{-k} \cap C_{-k'}$ consists of at most one point.

By replacing $\mathscr{P}$ with $-\mathscr{P}$ if necessary, we may assume that $C_{k_1} \cap C_{k_2}$ consists of at most one point for some k_1, $k_2 \in \mathscr{P}$. Consider an element v in another intersection $C_k \cap C_{k'}$, for $k \neq k'$. If $u = v - (v, k_2)\, k_2$, we have $(u, u) \leqslant 0$ and $(u, k_2) = 0$, which implies that $(u, k_1) = (v, k_1 - (k_1, k_2)\, k_2) \leqslant 0$. The element $w = k_1 - (k_1, k_2)\, k_2$ satisfies $(w, w) \leqslant 0$ and $w \sim v$ since $(v, w) \leqslant 0$; therefore a positive multiple w' of w belongs to $\mathscr{H}$. We have $(w', k) \leqslant 0$ and $(w', k') \leqslant 0$, at least one of the inequalities being strict. By the initial observation, v must be a multiple of $k + k'$ and $(k, k') = -1$.

(b) $\Rightarrow$ (a): The subspace spanned by distinct k and k' cannot be positive definite since C_k and $C_{k'}$ would then intersect at an interior point of $\mathscr{H}$; hence $|(k, k')| \geqslant 1$. If $(k, k') \geqslant 1$, one of the sets $C_k \cap C_{-k'}$, $C_{-k} \cap C_{k'}$

has at most one point, say $C_k \cap C_{-k'}$. Since $C_k \cap C_{k'}$ is also assumed to have at most one point, and their union is C_k, we obtain a contradiction. ∎

Suppose that W is a hyperbolic Coxeter group realised in V by Eqs. (1.1) and let

$$\Omega_r = \bigcup_{\omega_i \text{ real}} W\bar{\omega}_i \tag{3.2}$$

be the set of normalised real weights of W (where $\bar{\omega}_i = \omega_i/(\omega_i, \omega_i)^{1/2}$).

THEOREM 3.2. *Ω_r is a packing if and only if Γ is of level 2.*

Proof. This follows from Theorem 1.9 and the observation that the subspace spanned by a nonreal weight ω_i and another weight ω_j cannot be positive definite. ∎

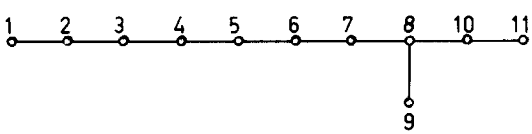
A disconnected level 2 graph gives rise to a trivial packing; we shall therefore restrict our attention to connected graphs only.

Let $\mathscr{E}_r$ be the convex closure of Ω_r; clearly $\mathscr{E}_r$ is contained in $\mathscr{E}$.

THEOREM 3.3. *Suppose that Γ is of level 2 and $\bar{\mathscr{E}}_r = \bar{\mathscr{E}}$. Then Ω_r is a maximal packing.*

Proof. A sphere which avoided all spheres corresponding to elements of Ω_r would correspond to a vector k, with $(k, k) = 1$ and $(k, \omega) \leqslant 0$ for all $\omega \in \Omega_r$. If $\bar{\mathscr{E}}_r = \bar{\mathscr{E}}$, we then have $(k, t) \leqslant 0$ for all $t \in \mathscr{E}$ and therefore $(k, k) \leqslant 0$ by Proposition 1.2, a contradiction. ∎

In view of Corollary 1.3, it seems plausible that $\bar{\mathscr{E}}_r = \bar{\mathscr{E}}$ is true for all hyperbolic Γ of level $\geqslant 2$. This can often be verified in an ad hoc manner; for instance, if Γ is the graph



only ω_1 is real, but one observes that

$$\omega_{i+1} = \omega_i + s_i \cdots s_1 \omega_i \qquad (1 \leqslant i \leqslant 7),$$

$$\omega_{11} = s_{10} s_8 \cdots s_1 \omega_1 + s_9 \cdots s_1 \omega_1,$$

$$\omega_9 = \omega_{11} + s_{11} s_{10} s_8 \cdots s_1 \omega_1, \omega_{10} = \omega_{11} + s_{11} \omega_{11}.$$

However, the author does not know if this holds for all graphs of level 2.

4. Equivalence of Packings

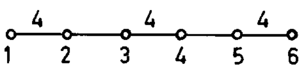
Packings $\mathscr{P}$ and $\mathscr{P}'$ are called *equivalent* if there exists an element $f \in O_l(V)$ such that $f(\mathscr{P}) = \mathscr{P}'$. The corresponding element of the conformal group then maps the set of spheres $\{S_k \mid k \in \mathscr{P}\}$ to the set $\{S_k \mid k \in \mathscr{P}'\}$.

The packing $\mathscr{P}$ is said to be of *lattice type* if the set $\mathbb{Z}\mathscr{P}$ of integral linear combinations of elements of $\mathscr{P}$ is a lattice. Equivalent packings of this kind have isomorphic lattices; in particular, the number

$$\delta = -2^{-N} \det(\mathbb{Z}\mathscr{P}) \tag{4.1}$$

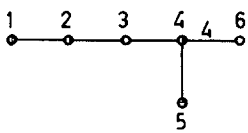
is an invariant of $\mathscr{P}$.

For a packing of the form (3.2), the set $\mathbb{Z}\Omega_r$ is clearly invariant under W. Hence Ω_r can be of lattice type only if W is crystallographic (i.e., leaves some lattice invariant). However, the converse need not be true; for example, if Γ is the graph



Ω_r contains both ω_1 and $2\sqrt{2}\,\omega_1$. The calculation of δ is illustrated by

Example 4.1. Suppose Γ is the graph



Only ω_1 is real and $\bar{\omega}_1 = \omega_1$. Using the formula $s_i(\omega_i) = -\omega_i + \sum_{j \neq i} 2(e_i, e_j)\,\omega_j$, we conclude in turn that $\mathbb{Z}\Omega_r$ must contain ω_2, ω_3, ω_4 and $\omega_5 \pm \sqrt{2}\,\omega_6$. Conversely, the lattice spanned by these vectors is invariant under W and must therefore be equal to $\mathbb{Z}\Omega_r$. Since $\det((e_i, e_j)) = -2^{-4}$, we have $\det((\omega_i, \omega_j)) = -2^4$, so that the determinant of the lattice spanned by ω_1, ω_2, ω_3, ω_4, ω_5 and $\sqrt{2}\omega_6$ is $(\sqrt{2})^2\,(-2^4) = -2^5$. As $\mathbb{Z}\Omega_r$ is of index 2 in this lattice, $\det(\mathbb{Z}\Omega_r) = 2^2(-2^5) = -2^7$ and $\delta = 2$.

An invariant applicable to any packing $\mathscr{P}$ is the number

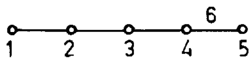
$$\nu = \sup\{(k, k') \mid k, k' \in \mathscr{P}, k \neq k'\}. \tag{4.2}$$

We have $\nu \leqslant -1$, with equality if two spheres corresponding to elements of $\mathscr{P}$ are in contact. For a packing of the form Ω_r, it is easy to see that

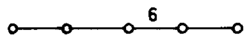
$$\nu = \max\{1 - 2/(\omega_i, \omega_i),\ (\bar{\omega}_i, \bar{\omega}_j) \mid \omega_i, \omega_j \text{ real}, i \neq j\}.$$

If Γ is nonstrict, then $\Gamma - i - j$ is euclidean for some $i \neq j$; both ω_i and ω_j must be real and span a singular plane, so that $(\bar{\omega}_i, \bar{\omega}_j) = -1$ and hence $\nu = -1$.

It frequently happens that a level 2 group contains a subgroup of the same kind in such a way that the set Ω_r are identical for both groups. Consider for instance the group W with the graph



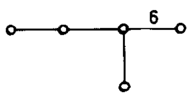
The subgroup W' generated by s_1, s_2, s_3, $s_4 s_5 s_4$, s_5 has $\{e_1, e_2, e_3, e_5 + \sqrt{3}e_4, e_5\}$ and $\{\omega_1, \omega_2, \omega_3, \omega_4/\sqrt{3}, \omega_5 - \omega_4/\sqrt{3}\}$ as its fundamental roots and weights, and corresponds to the graph



The only real weight of W is ω_1, whereas the real weights of W' are ω_1 and $\omega_5 - \omega_4/\sqrt{3} = s_4 s_3 s_2 s_1(\omega_1)/\sqrt{3}$, so that $\Omega_r' \subset \Omega_r$. Conversely, one readily sees that

$$W = W' \cup W's_4 \cup W's_4 s_3 \cup W's_4 s_3 s_2 \cup W's_4 s_3 s_2 s_1,$$

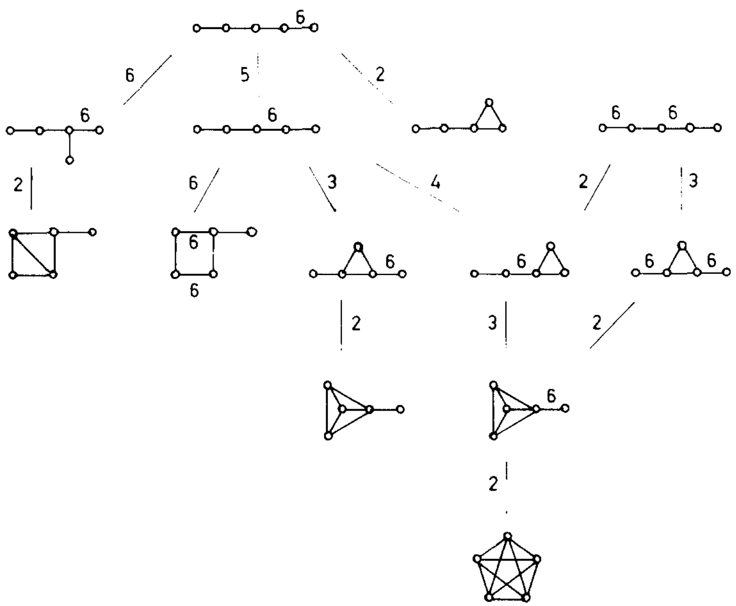
from which follows that $\Omega_r \subset \Omega_r'$. A similar situation prevails for the subgroup generated by s_1, s_2, s_3, $s_4 s_5 s_4$ and $s_5 s_4 s_5$, whose graph is



In general, whenever an edge marked by 4 or 6 occurs in the graph of W, one can apply this method to produce a subgroup. However, this subgroup need not be of level 2, nor do the weights have to behave in such a way as to produce identical packings. In the present case, all the graphs listed in Table I produce the same packing, namely, the three-dimensional analogue of the "Apollonian" packing of circles, first considered by Boyd [2]. In another instance, namely, the first graph listed under $N = 7$ in Table II, 17 distinct level 2 graphs produce the same packing.

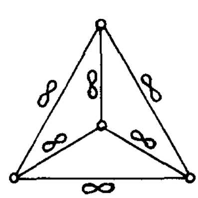
We have also indicated in Table I the indices of the various subgroups. Note that the cone $\mathscr{C}$ for the lowest group in the table admits the symmetric group S_5 as a group of symmetries. The semidirect product of S_5 with this group is isomorphic to our original group W. Using the high symmetry of this situation, the general algorithm for calculating elements of Ω, sketched in Section 1, can be refined to an algorithm of Boyd [4].

TABLE I



5. Classification of Packings

A graph with three vertices is of level 2 precisely if it contains a dotted edge, and all such graphs are strict. On the other hand, a graph with four vertices is of level 2 if and only if it contains no dotted edges and is not of level $\leqslant 1$, being strict if no edge is marked by ∞. For instance, the graph



corresponds to the "Apollonian" packing of circles discussed in the Introduction.

For $N \geqslant 5$, the number of level 2 graphs is finite. They can be derived by starting with the list of level 1 graphs in [5]; this has been done manually by the author, although a computer verification along the lines of [5] would be desirable. Every edge in such a graph is solid and marked by 3, 4, 5 or 6.

TABLE II

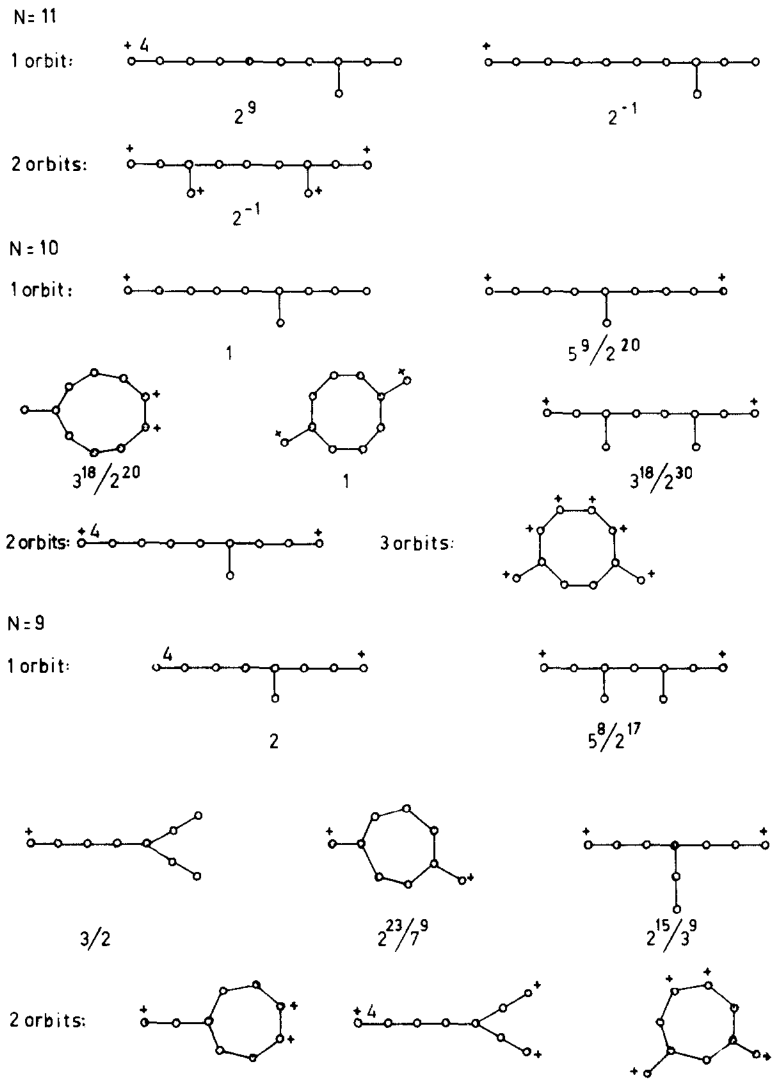

N = 11
1 orbit:
2^9
2^-1
2 orbits:
2^-1
N = 10
1 orbit:
1
5^9/2^20
3^18/2^20
1
3^18/2^30
2 orbits:
3 orbits:
N = 9
1 orbit:
2
5^8/2^17
3/2
2^23/7^9
2^15/3^9
2 orbits:

TABLE II (cont.)

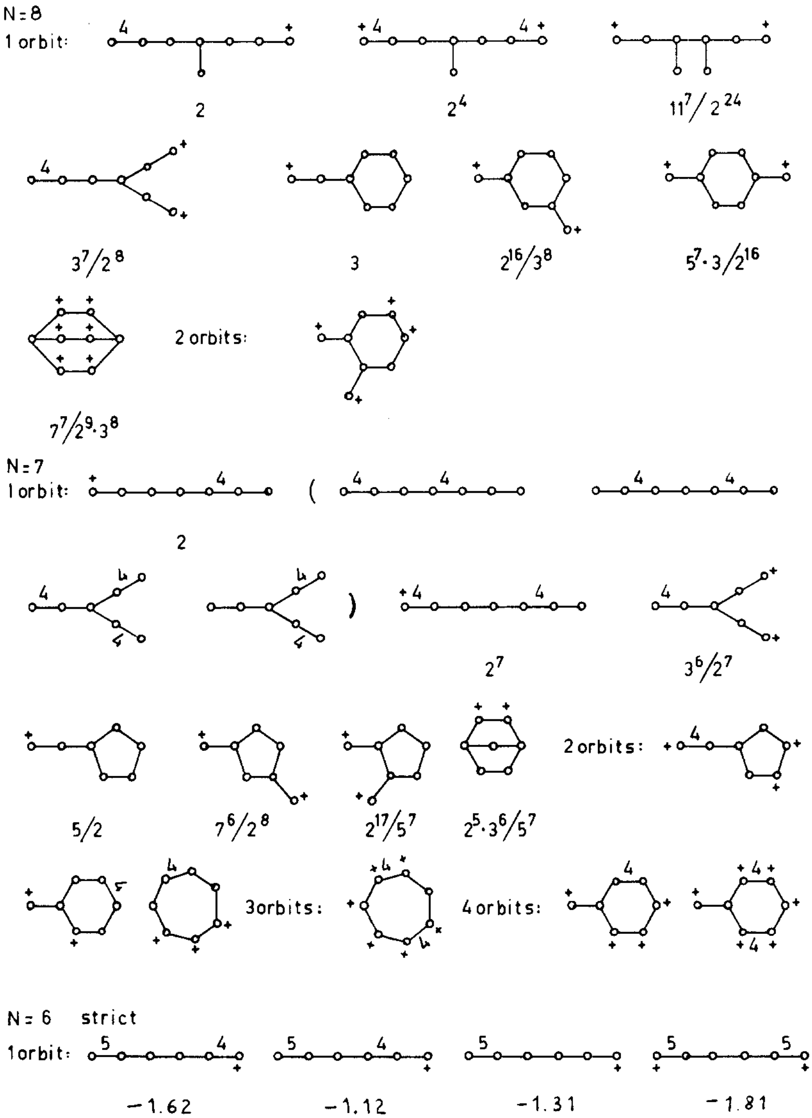


TABLE II (cont.)

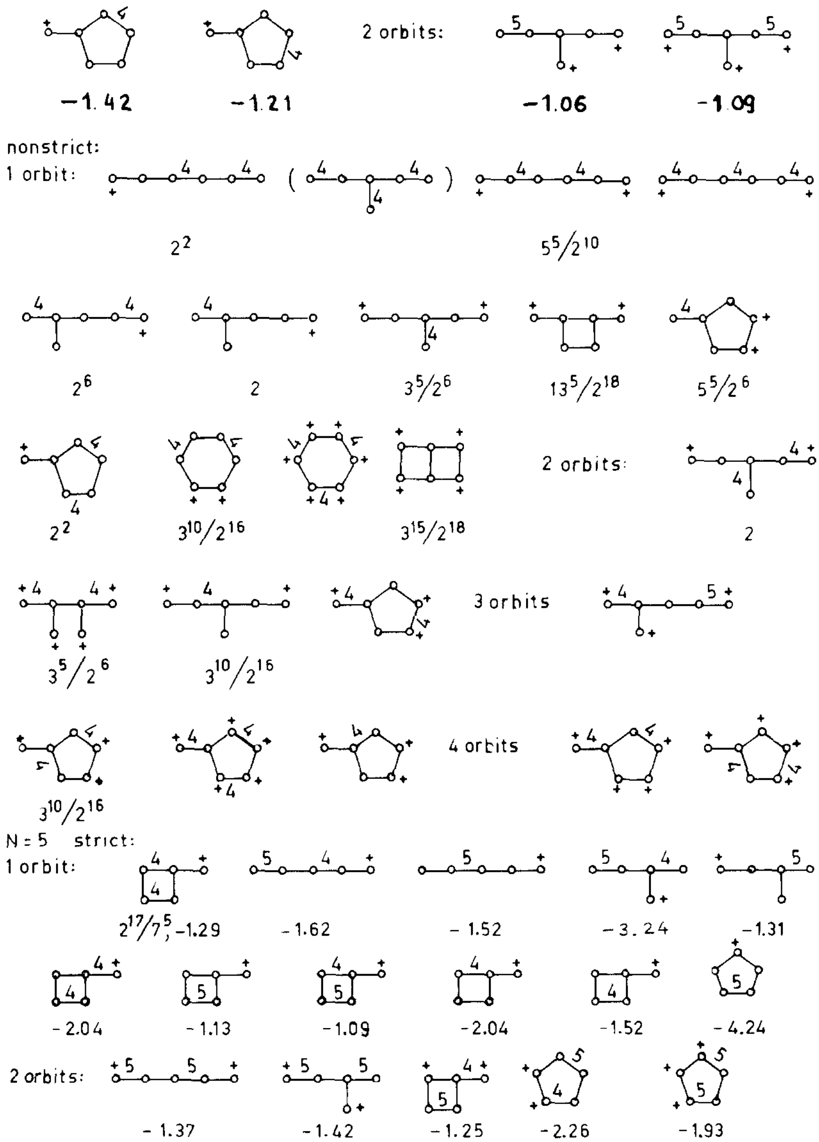


TABLE II (cont.)

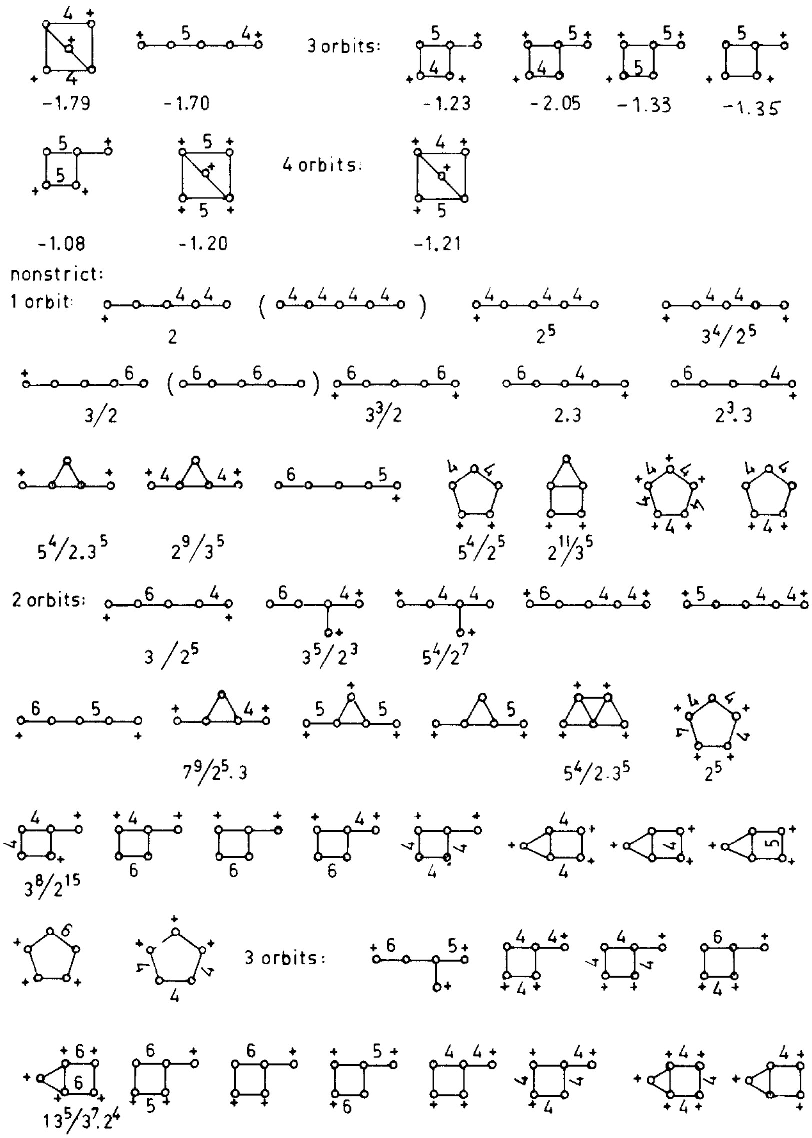
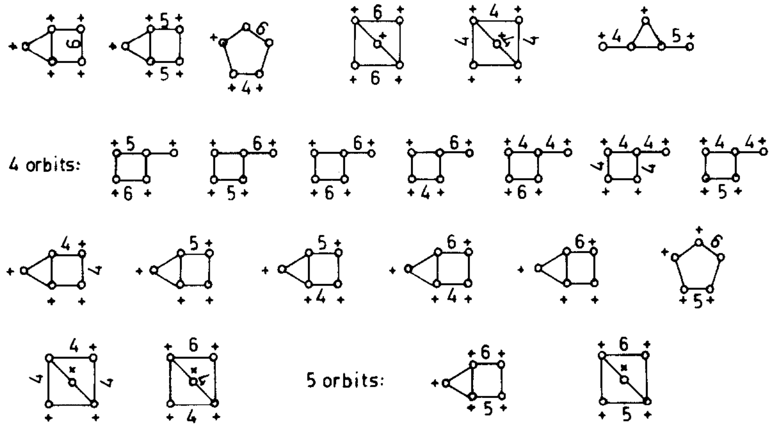


TABLE II (cont.)



Since our primary interest is in packings, we shall only list in Table II maximal elements from each family of groups yielding the same packing. In the four cases when there is more than one maximal element, we choose one and give the remaining ones in brackets immediately following. It is thus possible to recover the complete list of level 2 graphs by constructing subgroups by the method sketched in Section 4. For packings of lattice type, we indicate the invariant δ as a positive rational number underneath the graph. For strict packings, only occurring for $N = 5, 6$, we indicate the value of ν as a negative decimal. These invariants frequently show that the corresponding packings are inequivalent, but there are still many cases in Table II which could turn out to be equivalent. The numbers of packings and graphs obtained are as follows:

N	5	6	7	8	9	10	11
Graphs	186	66	36	13	10	8	4
Packings	95	30	13	9	8	7	3

Vertices $i \in \Gamma$ for which ω_i is real are marked with a "+."

Let G be the group of graph automorphisms of Γ, and let G act on V by $g \cdot e_i = e_{g(i)}$; then G normalises W. The group $\hat{W} = GW$ is a symmetry group of the packing Ω_r. We say that real weights ω_i and ω_j belong to the same orbit if $j = g(i)$ for some $g \in G$, and list the packings in Table II according to the number of orbits they contain.

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